Reaching for Understanding:

A Case Study

George Greenstein

Amherst College

ABSTRACT

When students transition from taking courses to doing research, they are switching from well-delineated projects to a more messy, fluid, back-and-forth world. How can we teach them the skills required of this new activity? A good pedagogical technique is to ask our students to watch us as we conduct research. Here I trace my own experiences as I reached for an understanding of a scientific problem, and I enumerate some general insights into the process of research.

1. INTRODUCTION

A crucial juncture is reached when students are ready to begin research. At this point, the very nature of what is asked of them changes. Previously they have been doing homework, lab exercises and the like. These are well-delineated projects, with well-defined goals and clearly stated methods as to how to proceed.

But beyond the classroom, life is different. The work of the practicing scientist is messy, fluid, back-and-forth, filled with blind alleys and false starts. Before this juncture in their education, students have been upon waters which have been charted for them. But now they are venturing into waters unknown – unknown even to their mentors.

How can we teach them the skills required to become navigators? How can we, who so far have focused our efforts on teaching them all the techniques of a scientific field, now switch gears and help them as they leave the safe harbors to which they have become accustomed, and start to use their new-found knowledge as they embark on research?

It seems to me that one good pedagogical technique is to ask our students to watch us as we navigate. By now we have, presumably, become at least somewhat adept at research: perhaps, as they observe us finding our way, they may pick up some useful insights.

In this regard, case studies can be helpful. In this essay I wish to trace, as well as I am able, my own experiences as I reached for an understanding of a scientific problem. It is an all-too human story I propose to tell, full of twists and turns, and I do not come across nearly as all-knowing and wise as I would like to appear. On the other hand, my story has at least the virtue of being the truth. My hope is that readers will profit from it.

Who are these intended readers? They fall into two camps: on the one hand they are those just starting out, either in an REU program or at the beginning of graduate dissertation research. On the other hand, they are those students’ advisors. Perhaps, in witnessing my stumbling progress, both will find insights to sustain beginning researchers as they set forth on their paths[[1]](#footnote-1).

In this account, I feel it is important to be specific, and to describe in detail the issues I was facing. So in what follows, even though my primary aim is to explicate in the process I went through as I reached for understanding, I will be careful to do so in the context of the actual problem before me. So expect to find a certain duality in this account: it is both a scientific analysis and a personal memoir. I am convinced that this approach is essential. Otherwise the account would amount to little more than platitudes and clichés. Only in the actual grappling with the issues does the full nature of research become evident. Sometimes I say to myself “There is no such thing as being smart in the abstract – there is only being smart about some specific thing.”

Remarkably, even though the particular focus of the work I am about to describe is quite modest, the general insights about the process of research are relevant in far more consequential realms. These insights will be in *italics.*

II. THE PROBLEM: THE VIRIAL THEOREM

One day, while more or less idly reading through a textbook, I came across a result that gripped my attention. Consider a group of objects exerting gravitational forces upon each other. There can be any number of them. Perhaps they are all the myriad bodies making up the Solar System. Or perhaps they are stars orbiting in a cluster or a galaxy. In simple cases it is possible to find the exact solution describing their motion – the two-body problem for example. But in more complicated cases it is not.

Nevertheless, one can prove the following. Suppose that somehow you reach in and give one of those objects a shove. You have sped it up. Thereafter it continues orbiting, interacting with all the other bodies in complicated ways. But if you compute the average kinetic energy of the entire assemblage over the long run, you will find that it has decreased. By increasing the kinetic energy of one object at one moment, you have decreased the net kinetic energy of all the objects over the long term.

This surprising result is based on the Virial Theorem[[2]](#endnote-1),[[3]](#endnote-2). “Slowing down by speeding up” is how I described it to myself.

*I always look for some vivid, simple phrase describing what I am working on. It helps me to concentrate on the nub of an issue, and not be distracted by the details. (It is also good for impressing friends at parties.)*

*The same is true in more consequential realms, The solution to the field equations of general relativity describing a single point mass was initially known as “the Schwarzschild solution” – but when the term “black Hole” was coined a watershed was passed*. T*he phrase “double helix” -- and the image – gripped the attention of public and geneticists alike.* *Many colleagues have mentioned to me that their fascination with science had been influenced by similarly vivid images or terminology.*

III. FINDING THE RIGHT MODEL

Even now, years later, I still recall my amazement at this surprising result. But I did nothing about it, because . . . well, because I was busy with other things.

That was long in the past. Perhaps a year ago, however, I found my thoughts returning to that strange theorem – and this time I decided to try to understand it. First I worked through the Virial Theorem’s proof: it was not so hard, and soon I felt I understood the mathematics. But still I found myself unsatisfied. So I decided to try to understand things in intuitive terms. And this meant that I started looking for some vivid, specific picture which I could hold in my mind. I don’t think in terms of mathematics – I think in terms of stuff that I can visualize.

*The Virial Theorem is elegant and exact, and its proof is straightforward. That is enough for many people. But it was not enough for me. Some people are of the first sort, others of the second. Each of us must come to a judgment of what counts as “understanding.”*

I decided that I needed to do a thought experiment. I needed to find a situation which embodied this strange result, and I needed to analyze it in full. Somewhere in the analysis, I figured, would lurk the essential insight that would satisfy me. This meant that my thought experiment had to be one which I could analyze it in detail without getting lost in the details. And that meant it had to be simple.

*Simplification is essential if one’s goal is to reach a general understanding. Often students want to flex their muscles and do everything in full complexity. Sometimes it is important to get them to calm down. An essential skill is that of deciding which details are important and which are not. One needs to be able to distinguish the forest from the trees.*

*All this is true if one is reaching a general understanding, as I was – or if one is writing a textbook, or teaching a course, or preparing a public lecture. But if the goal is full research, precisely the opposite is the case. In research, the details are essential. History is rife with instances in which people initially thought they understood a situation, but after diving into all the messy details they realized that they had not.*

My first thought experiment dealt with an orbiting spacecraft which fires its engine in the direction of its motion. Its velocity has increased. Suppose that spacecraft is initially in a circular orbit. After the engine fires, the orbit has become an ellipse: its current location is the perigee. A plot of velocity as a function of time would show an initially constant followed by a step upwards, then a gradual decrease to a minimum as the spacecraft rises upwards to apogee, and then an increase as it continues orbiting back towards its initial position. Apparently the average velocity – or at least the average velocity squared – along this new plot is less than its initial value.

Why does the spacecraft slow? Initially I thought in terms of Kepler’s Law of Areal Velocities. Later I found myself thinking in more intuitive terms: when the spacecraft is receding from Earth it is doing what a bicyclist does who stops pedaling and just coasts up a hill. Both are fighting gravity and therefore slow down. For some reason I found this second insight more satisfying.

*Again: what counts for “understanding,” at least in my case, is not so much mathematical as experiential.*

I felt I had found the bare bones of an insight: the phenomenon I was investigating had to do with how an orbiting body trades energy back and forth between kinetic and potential. I needed to put some flesh on those bones. Wanting to make my analysis rigorous, I dove into the subject of orbits. But the more I learned about them the more I realized how complicated they were. My simple thought experiment was turning into something not so simple.

*Moral: The model must one that can be analyzed in detail without getting lost in the details.*

Eventually – and reluctantly -- I backtracked. It took me a good deal of time, but finally I came up with a model so simple that it was almost laughable: the act of tossing a ball upwards. If I pretended that there was no air to resist the motion and the Earth was not rotating, the motion would be one-dimensional and the model would be simple enough that I could work out all the details without getting bogged down.

*Looking back on the whole process, it strikes me that this was the most important step: finding the right model to analyze.* *My main task was not to solve a problem -- it was to find the right problem to solve.*

The result I obtained is that for gentle tosses, if the initial velocity of the toss is increased, the average kinetic energy over the whole motion is increased as well. (Appendix I).

IV. FASTER TOSS

Next I turned to a second case in which an exact solution was available: a far more vigorous toss – one at escape velocity. The result I obtained is that a toss at escape velocity has an average kinetic energy of zero (Appendix II).

So now I was faced with an interesting situation. On the back of an envelope I sketched a crude graph: the horizontal axis was the velocity of toss, and the vertical axis the average kinetic energy. For slow velocities the curve led upwards – and yet for a certain large velocity it had the value zero. So for some velocity of toss, the upward-moving curve had to bend over into a downward trend – and this regime agreed with the Virial Theorem: speeding up led to a slowing down. In this regime was the understanding I was looking for. I needed to fill in the gap between my two exact solutions.

V. WANDERING

First I thought of things intuitively.

I pictured two balls: one labelled “S” which I tossed slowly, and one labelled “R” which I tossed a bit more rapidly. I considered the state just when the “R” ball had reached to top of its motion. Its kinetic energy was zero. But at this moment the “S” ball had already reached its peak and now was falling downwards with some kinetic energy. The two balls had exchanged their status: what once was faster was now slower.

I worked out details – lots of details. What fraction of the time was one ball moving faster than the other? I considered a particular point in space: as the two balls passed that point they had the same potential energy, but which had the greater kinetic energy?

Nothing much came of all this. I got some results, but nothing that led me to any great insight. Ultimately I changed gears.

*When should you abandon a line of work and switch to a new approach? More generally, how does one ever decide that enough is enough? There is no clear answer.*

*Consider the case of Subrahmanyan Chandrasekhar[[4]](#endnote-3). As a young man he had found that white dwarf stars beyond a certain mass cannot exist. That mass is now known as the Chandrasekhar Limit, and his work has entered the mainstream of astrophysics. But at the time, many senior and highly respected experts did not accept this conclusion.*

*Chandrasekhar argued for years for his result’s validity. The opposition did not let up. So eventually Chandrasekhar decided to leave the fray. He wrote up his result in the most rigorous and persuasive form, and then he moved on to other things.*

*Consider in contrast the case of Bill Borucki[[5]](#endnote-4), principal investigator of NASA’s Kepler spacecraft. This spacecraft was designed to detect extra-solar planets by observing the minute dips in a star’s brightness when an orbiting planet passes in front of it. Borucki first published a paper analyzing the problem in 1984. In 1992 he submitted a proposal to NASA: it was rejected. He worked to meet the reviewers’ objections and re-submitted two years later. The new proposal was rejected. Borucki kept at it until 2001, for a total of five proposals, before the project was finally accepted. The Kepler spacecraft was launched in 2009 – 25 years after he had begun his work.*

*The Kepler mission has been spectacularly successful, and Borucki is widely regarded as a hero. But suppose the project had never been funded? Suppose it had been funded, but failed once in orbit? Borucki had gambled a significant fraction of his entire career: only in retrospect did he learn that his bet had paid off.*

*Finally, consider Albert Einstein, who spent decades working to develop a unified field theory and never succeeded. What would this extraordinary person have achieved had he abandoned the project and turned his attention to other things?*

*Trial and error is the way research proceeds. The decision to keep on pushing in a given direction may or not be the right decision. There is no right answer: all one can do is make the choice and live with the consequences.*

Returning to my (far less significant!) case, I ultimately decided that intuition was not being very helpful. I headed off in a different direction. My previous results were valid for only certain velocities of toss. But a solution for a toss at any velocity was available, so I decided to just plod ahead and concentrate on that. The mathematics knew what was going on: it would tell me. (Details are given in Appendix III.)

First I chose a velocity of toss. The solution told me the kinetic energy at a sequence of times along the ball’s resulting motion. From these I found their averages. Then I did it again for a different toss velocity and so on. Figure 1 graphs the result: the average kinetic energy over the ball’s path as a function of the velocity with which it is tossed.



Fig. 1 Average kinetic energy (normalized to the initial potential energy) for various launch velocities.

*A comment about the vertical axis of Fig. 1. At first I simply plotted the average kinetic energy. But I worried that the particular numerical values depended on the mass and radius of the Earth, and the mass of my ball. I wanted something less parochial and more universal. Only after some fiddling did I find that, if I analyzed kinetic energy divided by the initial potential energy, all these irrelevant details factored out, and I was left with something that reflected the pure underlying physics. Similar comments apply to the horizontal axis.*

*Moral: It matters how we present our results. Even so “simple” a matter as the axes and caption on a graph can be significant.*

Fig. 1 showed exactly the behavior I had sketched on the back of my envelope. For high toss velocities, increasing the toss velocity decreased the average kinetic energy -- just what had so surprised me in the first place. And the parametric solution gave me the tool to understand it.

I did so by exploring the solution more fully. Something about the way the velocity varied along the ball’s path would allow me to understand things. Fig. 2 studies the velocity as a function of time for different velocities of toss.

Concentrate first on the slowest toss. The graph of velocity as a function of time is a straight line. A faster toss, though, reveals unexpected behavior. As the toss velocity is increased the curves start to show an inflection point.

“Inflection point” is a perfectly good term of mathematics. But had I left things at that I would never have progressed further in my understanding. The crucial step I took was the next one, which was to think of those inflections in a more intuitive way. And the remarkable thing was the reason I did so: I tried to explain my conundrum to a non-scientist.

I was therefore forced to speak in different terms. I opened my mouth and heard myself say “Those graphs are telling me that my rapidly-tossed ball is spending a good deal of time close to the zero-velocity line . . . which is a fancy way of saying that it is moving slowly for a surprisingly great amount of time.” And as I heard myself say this, I realized what it meant: for a high toss velocity, the ball had low kinetic energy for a great fraction of its period.

*Moral: think about things in as many different ways as possible – and speak about them to as many different kinds of people as possible. The more you re-phrase your problem in different terms, the more opportunities you will find to move beyond your current understanding.*



Fig. 2 Time evolution of the velocity for various launch velocities.

My experience with my friend led me to think visually. How did those curves tell me to imagine the ball’s motion? Gazing at Fig. 2, I saw in my mind’s eye a gently tossed ball rising upwards, reaching the top of its path, quickly reversing and then accelerating downwards. But the rapidly-tossed ball reversed only slowly at the top of its path -- it seemed to dawdle through its low-velocity state for a surprisingly great length of time.

*Moral: for me at least, visual thinking is vital. I need to find some way of imagining things –in this case, to make the abstract world of the Virial Theorem vivid. Many theoretical physicists are quite different: for them, it is the mathematics that is vital. Which are you?*

Thinking in such terms led me to ask a new question: What was so special about the top of its path to my ball’s average behavior? Fig. 3 gave me the answer. It graphs the fractional change in the height of the ball throughout its motion. I saw that for gentle tosses the ball’s distance from its starting point is essentially constant -- the starting distance is the radius of the Earth: the maximum distance is that radius plus a few meters. But for rapid tosses it is not. Indeed, for the fastest toss the ball has risen to 4.3 times its initial height at the top of its motion.



Fig. 3 Fractional change in height over the course of the motion for various toss velocities.

But gravity is an inverse square force. At the peak of this ball’s motion, I realized, gravity is (4.3)2 = 18.5 times weaker than at its start.

Finally I understood my ball’s strange behavior. A rapidly-tossed ball has so much kinetic energy that it rises into regions where gravity is weak. So just where the kinetic energy is the least is where the rate of change of that energy the least.

I had reached my insight. Finally I felt that I understood “slowing down by speeding up.”

VI. A GLITCH

My satisfaction was short-lived. Almost immediately I realized there was a problem.

The problem was that the Virial Theorem does not apply to my model. The Theorem requires the motion to be periodic, but mine was not: while at the start the ball was moving up, at the end it was moving down. A further issue is that even if the motion is not periodic, so long as it remains bounded the Theorem applies – but only over the long term, and my motion was short-term.

So why did I persist in studying such an inappropriate case? Partially it was a matter of hope: once the ball came back down I could always allow it to bounce and so return to its initial state, and I figured that some day I would find a way to incorporate the bounce into my analysis. But there was a more important reason driving my choice: the fact that I already knew that for rapid tosses my model exhibited the same counterintuitive behavior that had gripped me in the first place. So I had plenty of reason to continue.

But to be honest, as I look backwards on my path I believe that most of all I was motivated by a quotation I had come across many years earlier. Someone had discovered a strange new phenomenon: asked what he thought he had discovered, he responded “I did not think. I investigated.” And I did too -- I just kept plodding on.

And I am glad, for quite apart from my initial motivation I ended up showing that “slowing down by speeding up” occurs in wider realms than had been previously thought. The Virial Theorem shows that it occurs in certain circumstances: I had shown that it occurs in other circumstances as well. Earth-shaking? No. But interesting? Well, yes.

*Moral: the paths of understanding are winding and various.*

Is it possible to extend my model to cover the fully periodic motion that the Virial Theorem requires? Perhaps it is. I’m working on it right now, in fact.

*Science is never over. Questions are seldom answered perfectly and fully, with no further research required. Furthermore, even after a conclusion has been reached, its ultimate fate must await validation or rejection through the work of others. No matter how far you have travelled, there is always further to go.*

*A few quotations – and these are actual statements by working physicists:*

* *“We thought we had things pretty well figured out. But then we made a mistake: we did another experiment.”*
* *“The recent discovery of \_\_\_\_\_\_\_\_ has shed new darkness on the question of \_\_\_\_\_\_\_\_”*
* *“Back then I didn’t know enough to realize that I was wrong.”*

If I do succeed in this new project, I will write up my insight and send it off to be published. But in that publication I will take care to omit all the twists and turns that I have recounted here. Rather I will present my new understanding in the best possible light: as a perfect argument, proceeding from first principles in a straight line to a clean, unassailable conclusion – the ideal of scientific reasoning.

That is how science is presented. It is how we learned it from our textbooks, and it is how we publish our research. And it is how the public perceives scientists: as dispassionate and perfect reasoners. But it is not how research is actually performed.

APPENDIX I

SLOW TOSS

A ball thrown vertically upwards with a modest initial velocity moves with constant acceleration

(1)

where g is the acceleration of gravity and the ball’s initial velocity. The highest point in its motion is reached when the velocity is zero: this occurs at a time

= / (2)

The period is twice

To find the average kinetic energy compute

(3)

A straightforward calculation yields

(4)

which shows that, if the initial velocity of the toss is increased, the average kinetic energy over the whole motion is increased.

APPENDIX II

TOSS AT ESCAPE VELOCITY

Consider a mass point M located at the origin attracting a test mass m located at position z. The ball’s total energy is

(5)

From Eq. (5) escape velocity (E=0) is

(6)

where is the position from which I toss the ball. One can solve for the motion in this case: setting E = 0

(7)

It can be verified by direct substitution that the solution to Eq. (7) is

(8)

The velocity as a function of time is then

=

where (9)

And the average kinetic energy between an initial time and a final time is

- ) / -

To find the average over the whole motion let greatly exceed in which case the average kinetic energy is proportional to and so approaches zero as increases without limit.

APPENDIX III

TOSS AT INTERMEDIATE VELOCITIES

Consider again a mass point M located at the origin attracting a test mass m located at position z. For nonzero total energy E we find from Eq. (5)

(11)

It can be verified by direct substitution that for negative E the parametric solution to Eq. (11) is

(12)

where ranges from to and

(13)

Similarly the velocity is given by

(14)

This solution can easily be explored numerically. Choose values for the constants appearing in and . Plug in a value for , compute the time, position and velocity; advance to a new value; repeat the process and accumulate the results in a table. Finally stop paying attention to the column of the table and concentrate on the times and velocities. From the velocity so obtained one can easily compute the required kinetic energy and then its average.

(There is one issue however that needs to be discussed. The choice leads to and an infinite velocity. This is not surprising since the model is of a mass point located at the origin attracting the ball: if is zero the gravitational force is infinite. In reality, of course, one starts the motion at some non-zero (the radius of the Earth, say). This leads to a certain non-zero value of the time. One then subtracts these two starting values from the and of Eq. (12) to find an elapsed distance and time: these are what are presented in the results to follow.)

REFERENCES

1. I should emphasize that in this essay I do not pretend to be doing science education research. Nothing here is as rigorous and well-defined as work in that field. Rather my goal is far more fluid and personal. [↑](#footnote-ref-1)
2. Herbert Goldstein, *Classical Mechanics* (Addison-Wesley, Reading MA, 1959), pp. 69-71. [↑](#endnote-ref-1)
3. Bradley W. Carroll and Dale A. Ostlie, *An Introduction to Modern Astrophysics, 2nd ed*. (Pearson Addison-Wesley, San Francisco, 2007), pp. 50-53.

   [↑](#endnote-ref-2)
4. Oral History Interview conducted by the American Institute of Physics

   <https://www.aip.org/history-programs/niels-bohr-library/oral-histories/browse/c> [↑](#endnote-ref-3)
5. [*https://servicetoamericamedals.org/honorees/william-j-borucki/*](https://servicetoamericamedals.org/honorees/william-j-borucki/) [↑](#endnote-ref-4)